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# TWO-PARAMETER QUANTUM GROUPS AND RINGEL-HALL ALGEBRAS OF $A_\infty$ -TYPE

XIN TANG

**ABSTRACT.** In this paper, we study the two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$  associated to the Lie algebra  $\mathfrak{sl}_\infty$  of infinite rank. We shall prove that the two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$  admits both a Hopf algebra structure and a triangular decomposition. In particular, it can be realized as the Drinfeld double of its certain Hopf subalgebras. We will also study a two-parameter twisted Ringel-Hall algebra  $H_{r,s}(A_\infty)$  associated to the category of all finite dimensional representations of the infinite linear quiver  $A_\infty$ . In particular, we will establish an iterated skew polynomial presentation of  $H_{r,s}(A_\infty)$  and prove that  $H_{r,s}(A_\infty)$  is a direct limit of the directed system of the two-parameter Ringel-Hall algebras  $H_{r,s}(A_n)$  associated to the finite linear quiver  $A_n$ . As a result, we construct a PBW basis for  $H_{r,s}(A_\infty)$  and prove that all prime ideals of  $H_{r,s}(A_\infty)$  are completely prime. Furthermore, we will establish an algebra isomorphism from  $U_{r,s}^+(\mathfrak{sl}_\infty)$  to  $H_{r,s}(A_\infty)$ , which enable us to obtain the corresponding results for  $U_{r,s}^+(\mathfrak{sl}_\infty)$ . Finally, via the theory of generic extensions in the category of finite dimensional representations of  $A_\infty$ , we shall construct several monomial bases and a bar-invariant basis for  $U_{r,s}^+(\mathfrak{sl}_\infty)$ .

## INTRODUCTION

As generalizations or variations of the notation of quantum groups [13], several multi-parameter quantum groups have appeared in the literatures [1, 8, 11, 12, 16, 18, 25, 31, 32]. Let  $\mathfrak{g}$  be a finite dimensional complex simple Lie algebra. Let us choose  $r, s \in \mathbb{C}^*$  in such a way that  $r, s$  are transcendental over  $\mathbb{Q}$ . The study of the two-parameter quantum group  $U_{r,s}(\mathfrak{g})$  has been revitalized in [3, 4, 5, 6, 7] and the references therein. Note that the one-parameter quantum groups associated to Lie algebras  $\mathfrak{gl}_\infty, \mathfrak{sl}_\infty$  of infinite ranks [17] have been studied

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in the literatures [10, 19, 22, 23]. Similar to the case of one-parameter quantum groups, one might be interested in the constructions of the corresponding two-parameter quantum groups.

It is the purpose of this paper to study the two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$ , where the Lie algebra  $\mathfrak{sl}_\infty$  consists of all infinite trace-zero square matrices with only finitely many non-zero entries. We shall first define such a two-parameter quantum group and then study some of its basic properties. As a matter of fact, we will formulate the two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$  as a limit version of the two-parameter quantum groups  $U_{r,s}(\mathfrak{sl}_{n+1})$ ,  $n \geq 1$  as studied in [5]. As usual, we will prove that the algebra  $U_{r,s}(\mathfrak{sl}_\infty)$  admits a Hopf algebra structure and it is the Drinfeld double of its certain Hopf subalgebras.

To further investigate the structure of the two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$ , we shall study its subalgebra  $U_{r,s}^+(\mathfrak{sl}_\infty)$  employing the approach of Ringel-Hall algebras. Note that the Ringel-Hall algebra approach has found many important applications in the study of one-parameter quantum groups [15, 20, 21, 24, 26, 27, 28, 29, 30, 34] and the references therein. To this end, we shall first define a two-parameter twisted Ringel-Hall algebra  $H_{r,s}(A_\infty)$  associated to the category of all finite dimensional representations of the infinite linear quiver  $A_\infty$ . Then we shall prove that the algebra  $H_{r,s}(A_\infty)$  can be presented as an iterated skew polynomial ring, and thus construct a PBW basis for  $H_{r,s}(A_\infty)$ . Furthermore, we shall prove that  $H_{r,s}(A_\infty)$  is a direct limit of the two-parameter twisted Ringel-Hall algebras  $H_{r,s}(A_n)$ ,  $n \geq 1$  associated to the finite linear quivers  $A_n$ ,  $n \geq 1$  (See [24, 33]). As an application, we prove that all prime ideals of  $H_{r,s}(A_\infty)$  are indeed completely prime.

To transfer the information to the algebra  $U_{r,s}^+(\mathfrak{sl}_\infty)$ , we will establish an algebra isomorphism from  $U_{r,s}^+(\mathfrak{sl}_\infty)$  onto  $H_{r,s}(A_\infty)$ . On the one hand, such an algebra isomorphism provides a generator-relation presentation of the two-parameter Ringel-Hall algebra  $H_{r,s}(A_\infty)$ , which has been defined over a prescribed basis. On the other hand, via this isomorphism, we can prove that the algebra  $U_{r,s}^+(\mathfrak{sl}_\infty)$  can be presented as an iterated skew polynomial ring and it is a direct limit of  $U_{r,s}^+(\mathfrak{sl}_{n+1})$ ,  $n \geq 1$ . As a result, we are able to construct a PBW basis for  $U_{r,s}^+(\mathfrak{sl}_\infty)$  and prove that all prime ideals of  $U_{r,s}^+(\mathfrak{sl}_\infty)$  are completely prime.

To study the Borel subalgebras  $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$  (resp.  $U_{r,s}^{\leq 0}(\mathfrak{sl}_\infty)$ ) of the two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$ , we will define a Hopf algebra structure on the extended two-parameter twisted Ringel-Hall algebra  $\overline{H_{r,s}(A_\infty)}$  and establish an Hopf algebra isomorphism from the two-parameter quantized Hopf algebra  $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$  onto  $\overline{H_{r,s}(A_\infty)}$ . We will

follow the lines in [15, 34]. This result shall provide a realization of the whole two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$  via the double of two-parameter extended Ringel-Hall algebras.

Note that there exists a  $\mathbb{Q}$ -algebra automorphism (which will be called the bar-automorphism) on the algebra  $U_{r,s}^+(\mathfrak{sl}_\infty)$ , which exchanges  $r^{\pm 1}$  and  $s^{\pm 1}$  and fixes the generators  $e_i$ . Using the theory of generic extensions in the category of finite dimensional representations of  $A_\infty$ , we will construct several monomial bases for the two-parameter quantum groups following the idea used in [9, 24]. As an application, we will also construct a bar-invariant basis for the algebra  $U_{r,s}^+(\mathfrak{sl}_\infty)$  following [24].

The paper is organized as follows. In Section 1, we give the definition of  $U_{r,s}(\mathfrak{sl}_\infty)$  and study some of its basic properties. In Section 2, we define and study two-parameter Ringel-Hall algebra  $H_{r,s}(A_\infty)$  and establish the algebra isomorphism from  $U_{r,s}^+(\mathfrak{sl}_\infty)$  onto  $H_{r,s}(A_\infty)$ . In Section 3, we define and study the extended two-parameter Ringel-Hall algebra  $\overline{H_{r,s}(A_\infty)}$  and establish the Hopf algebra isomorphism from  $U_{r,s}^{\geq 0}(\mathfrak{g})$  onto  $\overline{H_{r,s}(A_\infty)}$ . In Section 4, we use generic extension theory to construct some monomial bases and a bar-invariant basis for  $U_{r,s}^+(\mathfrak{sl}_\infty)$ .

## 1. DEFINITION AND BASIC PROPERTIES OF THE TWO-PARAMETER QUANTUM GROUPS $U_{r,s}(\mathfrak{sl}_\infty)$

First of all, let us fix some notation. Let  $r, s$  be two-parameters chosen from  $\mathbb{C}^*$ , such that  $r, s$  are transcendental over the field  $\mathbb{Q}$  and  $r^m s^n = 1$  implies  $m = n = 0$ . Let us set  $\mathcal{Z} = \mathbb{Z}[r^{\pm 1}, s^{\pm 1}]$  and  $\mathcal{A} = \mathbb{Q}[r, s]_{(r-1, s-1)}$ , which is the localization of  $\mathbb{Q}[r, s]$  at the maximal ideal  $(r-1, s-1)$ .

Let  $\mathfrak{sl}_\infty$  denote the infinite dimensional complex Lie algebra which consists of all trace-zero square matrices  $(a_{ij})_{i,j \in \mathbb{N}}$  with only finitely many non-zero entries. The one-parameter quantum groups  $U_q(\mathfrak{sl}_\infty)$  associated to  $\mathfrak{sl}_\infty$  were studied by various people in the references [10, 19, 22, 23]. Following a similar idea in [5, 32], we will introduce a class of two-parameter quantum groups  $U_{r,s}(\mathfrak{sl}_\infty)$  associated to the Lie algebra  $\mathfrak{sl}_\infty$ .

It is well known that one can also define roots for the Lie algebra  $\mathfrak{sl}_\infty$  as in the finite dimensional case of  $\mathfrak{g} = \mathfrak{sl}_{n+1}$ . In particular, all the simple roots of  $\mathfrak{sl}_\infty$  can be denoted as  $\alpha_i, i \in I = \mathbb{N}$ . Accordingly, all the positive roots of  $\mathfrak{sl}_\infty$  are exactly given as  $\alpha_{ij} := \sum_{k=i}^j \alpha_k$  for  $i \leq j \in \mathbb{N}$ .

Let  $C = (c_{ij})_{i,j \in \mathbb{N}}$  denote the infinite Cartan matrix corresponding to the Lie algebra  $\mathfrak{sl}_\infty$ . Then, we have the following

$$c_{ii} = 2, c_{ij} = -1 \text{ for } |i - j| = 1, c_{ij} = 0 \text{ for } |i - j| > 1.$$

Let  $\mathbb{Q}(r, s)$  denote the function field in two variables  $r, s$  over the field  $\mathbb{Q}$  of all rational numbers. As mentioned early, we may also choose  $r, s$  to be complex numbers which are transcendental over  $\mathbb{Q}$  such that  $r^2 \neq s^2$ . In the future, more restrictions may be put on the parameters if needed. Let  $\mathcal{Q}$  denote the root lattice generated by  $\alpha_i, i \in \mathbb{N}$ . Then we can define a bilinear form  $\langle -, - \rangle$  on the root lattice  $\mathcal{Q} \cong \mathbb{Z}^{\oplus \mathbb{N}}$  as follows

$$\langle i, j \rangle := \langle \alpha_i, \alpha_j \rangle = \begin{cases} a_{ij}, & \text{if } i < j, \\ 1, & \text{if } i = j, \\ 0, & \text{if } i > j. \end{cases}$$

The above defined bilinear form is a generalization of the Ringel-Euler form in the finite case of  $\mathfrak{g} = \mathfrak{sl}_{n+1}$ . In terms of the above defined bilinear form  $\langle -, - \rangle$ , one can give the following definition of the two-parameter quantum groups  $U_{r,s}(\mathfrak{sl}_\infty)$  associated to the Lie algebra  $\mathfrak{sl}_\infty$ .

**Definition 1.1.** The two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$  is defined to be the  $\mathbb{Q}(r, s)$ -algebra generated by  $e_i, f_i, w_i^{\pm 1}, w_i'^{\pm 1}, i \in \mathbb{N}$  subject to the following relations

$$\begin{aligned} w_i^{\pm 1} w_j^{\pm 1} &= w_j^{\pm 1} w_i^{\pm 1}, & w_i^{\pm 1} w_j'^{\pm 1} &= w_j'^{\pm 1} w_i^{\pm 1}, \\ w_i^{\pm 1} w_j'^{\pm 1} &= w_j'^{\pm 1} w_i^{\pm 1}, & w_i^{\pm 1} w_i^{\mp 1} &= 1 = w_i'^{\pm 1} w_i'^{\mp 1}, \\ w_i e_j &= r^{\langle j, i \rangle} s^{-\langle i, j \rangle} e_j w_i, & w_i' e_j &= r^{-\langle i, j \rangle} s^{\langle j, i \rangle} e_j w_i', \\ w_i f_j &= r^{-\langle j, i \rangle} s^{\langle i, j \rangle} f_j w_i, & w_i' f_j &= r^{\langle i, j \rangle} s^{-\langle j, i \rangle} f_j w_i', \\ e_i f_j - f_j e_i &= \delta_{i,j} \frac{w_i - w_i'}{r_i - s_i}, \\ e_i e_j - e_j e_i &= f_i f_j - f_j f_i = 0 \text{ for } |i - j| > 1, \\ e_i^2 e_{i+1} - (r + s) e_i e_{i+1} e_i + r s e_{i+1} e_i^2 &= 0, \\ e_i e_{i+1}^2 - (r + s) e_{i+1} e_i e_{i+1} + r s e_{i+1}^2 e_i &= 0, \\ f_i^2 f_{i+1} - (r^{-1} + s^{-1}) f_i f_{i+1} f_i + (rs)^{-1} f_{i+1} f_i^2 &= 0, \\ f_i f_{i+1}^2 - (r^{-1} + s^{-1}) f_{i+1} f_i f_{i+1} + (rs)^{-1} f_{i+1}^2 f_i &= 0. \end{aligned}$$

First of all, we have the following obvious proposition concerning a Hopf algebra structure of the algebra  $U_{r,s}(\mathfrak{sl}_\infty)$ .

**Proposition 1.1.** *The algebra  $U_{r,s}(\mathfrak{g})$  is a Hopf algebra with the co-multiplication, counit and antipode defined as follows*

$$\begin{aligned}\Delta(w_i^{\pm 1}) &= w_i^{\pm 1} \otimes w_i^{\pm 1}, & \Delta(w_i'^{\pm 1}) &= w_i'^{\pm 1} \otimes w_i'^{\pm 1}, \\ \Delta(e_i) &= e_i \otimes 1 + w_i \otimes e_i, & \Delta(f_i) &= 1 \otimes f_i + f_i \otimes w_i', \\ \epsilon(w_i^{\pm 1}) &= \epsilon(w_i'^{\pm 1}) = 1, & \epsilon(e_i) &= \epsilon(f_i) = 0, \\ S(w_i^{\pm 1}) &= w_i^{\mp 1}, & S(w_i'^{\pm 1}) &= w_i'^{\mp 1}, \\ S(e_i) &= -w_i^{-1}e_i, & S(f_i) &= -f_iw_i'^{-1}.\end{aligned}$$

**Proof:** The proof is reduced to the finite case where  $\mathfrak{g} = \mathfrak{sl}_{n+1}$ , whose proof can be found in [5]. And we will not repeat the details here.  $\square$

Let  $U_{r,s}^+(\mathfrak{sl}_\infty)$  (resp.  $U_{r,s}^-(\mathfrak{sl}_\infty)$ ) denote the subalgebra of  $U_{r,s}(\mathfrak{sl}_\infty)$  generated by  $e_i, i \in \mathbb{N}$  (resp. by  $f_i, i \in \mathbb{N}$ ). Let  $U_{r,s}^0(\mathfrak{sl}_\infty)$  denote the subalgebra of  $U_{r,s}(\mathfrak{sl}_\infty)$  generated by  $w_i^{\pm 1}, w_i'^{\pm 1}, i \in \mathbb{N}$ . Then we shall have the following triangular decomposition of  $U_{r,s}(\mathfrak{sl}_\infty)$ .

**Proposition 1.2.** *The algebra  $U_{r,s}(\mathfrak{sl}_\infty)$  has a triangular decomposition*

$$U_{r,s}(\mathfrak{sl}_\infty) \cong U_{r,s}^-(\mathfrak{sl}_\infty) \otimes U_{r,s}^0(\mathfrak{sl}_\infty) \otimes U_{r,s}^+(\mathfrak{sl}_\infty).$$

**Proof:** Once again, we can repeat the proof used in the case of  $U_{r,s}(\mathfrak{sl}_{n+1})$ . We refer the reader to [5] for more details.  $\square$

Let us denote by  $\mathbb{Z}^{\oplus \mathbb{N}}$  the free abelian group of rank  $|\mathbb{N}|$  with a basis denoted by  $z_1, z_2, \dots, z_n, \dots$ . Given any element  $\mathbf{a} \in \mathbb{Z}^{\oplus \mathbb{N}}$ , say  $\mathbf{a} = \sum a_i z_i$ , we set  $|\mathbf{a}| = \sum a_i$ . Note that algebra  $U_{r,s}^+(\mathfrak{sl}_\infty)$  (resp.  $U_{r,s}^-(\mathfrak{sl}_\infty)$ ) is a  $\mathbb{Z}^{\oplus \mathbb{N}}$ -graded algebra by assigning to the generator  $e_i$  (resp.  $f_i$ ) the degree  $z_i$ . Given  $\mathbf{a} \in \mathbb{Z}^{\oplus \mathbb{N}}$ , we denote by  $U_{r,s}^\pm(\mathfrak{sl}_\infty)_{\mathbf{a}}$  the set of homogeneous elements of degree  $\mathbf{a}$  in  $U_{r,s}^\pm(\mathfrak{sl}_\infty)$ .

**Proposition 1.3.** *We have the following decomposition*

$$U_{r,s}^+(\mathfrak{sl}_\infty) = \bigoplus_{\mathbf{a}} U_{r,s}^+(\mathfrak{sl}_\infty)_{\mathbf{a}}, \quad U_{r,s}^-(\mathfrak{sl}_\infty) = \bigoplus_{\mathbf{a}} U_{r,s}^-(\mathfrak{sl}_\infty)_{\mathbf{a}}.$$

$\square$

Let us define  $U_{v,v^{-1}}(\mathfrak{sl}_\infty)$  to be the specialization of  $U_{r,s}(\mathfrak{sl}_\infty)$  for  $r = v = s^{-1}$ . Then we shall have the following similar result as [5], whose proof is exactly the same as the one in [5].

**Proposition 1.4.** *Assume there exists an isomorphism of Hopf algebras*

$$\phi: U_{r,s}(\mathfrak{sl}_\infty) \longrightarrow U_{v,v^{-1}}(\mathfrak{sl}_\infty)$$

*for some  $v$ . Then  $r = v$  and  $s = v^{-1}$ .*

$\square$

**1.1. A Drinfeld double realization of  $U_{r,s}(\mathfrak{sl}_\infty)$ .** In this subsection, we show that the two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$  can be realized as the Drinfeld double of its certain Hopf subalgebras. To proceed, we need to recall a couple of standard definitions for the Hopf pairing and the Drinfeld double of Hopf algebras. For more details about these concepts, we refer the reader to the references [5, 12].

**Definition 1.2.** A Hopf pairing of two Hopf algebras  $H'$  and  $H$  is a bilinear form  $(\cdot, \cdot): H' \times H \longrightarrow \mathbb{K}$  such that

$$(1) (1, h) = \epsilon_H(h),$$

$$(2) (h', 1) = \epsilon_{H'}(h'),$$

$$(3) (h', hk) = (\Delta_{H'}(h'), h \otimes k) = \sum (h'_{(1)}, h)(h'_{(2)}, k),$$

$$(4) (h'k', h) = (h' \otimes k', \Delta_H(h)) = \sum (h', h_{(1)})(k', h_{(2)}),$$

for all  $h, k \in H, h', k' \in H'$ , where  $\epsilon_H, \epsilon_{H'}$  denote the counits of  $H, H'$  respectively, and  $\Delta_H, \Delta_{H'}$  denote their comultiplications.

It is obvious that

$$(S_{H'}(h'), h) = (h', S_H(h))$$

for all  $h \in H$  and  $h' \in H'$ , where  $S_{H'}$  and  $S_H$  denote the respective antipodes of  $H$  and  $H'$ .

Let  $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$  (resp.  $U_{r,s}^{\leq 0}(\mathfrak{sl}_\infty)$ ) be the Hopf subalgebra of  $U_{r,s}(\mathfrak{sl}_\infty)$  generated by  $e_i, w_i^{\pm 1}$  (resp.  $f_i, w_i'^{\pm 1}$ ). Assume that  $B = U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$  and  $(B')^{coop}$  is the Hopf algebra generated by  $f_j, (w_j')^{\pm 1}$  with the opposite coproduct to  $U_{r,s}^{\leq 0}(\mathfrak{sl}_\infty)$ . Using the same proof in the case of  $\mathfrak{sl}_{n+1}$  [5], we shall have the following result

**Lemma 1.1.** *There exists a unique Hopf pairing  $B$  and  $B'$  such that*

$$(f_i, e_j) = \frac{\delta_{i,j}}{s-r}$$

$$(w_i', w_j) = r^{\langle e_i, e_j \rangle} s^{-\langle e_j, e_i \rangle},$$

and the pairing takes the zero value on all other pairs of generators. Moreover, we have  $(S(a), S(b)) = (a, b)$  for  $a \in B', b \in B$ .

□

Therefore, we have the following similar result as in [5].

**Theorem 1.1.**  *$U_{r,s}(\mathfrak{sl}_\infty)$  can be realized as a Drinfeld double of Hopf subalgebras  $B = U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$  and  $(B')^{coop} = U_{r,s}^{\leq 0}(\mathfrak{sl}_\infty)$ , that is,*

$$U_{r,s}(\mathfrak{sl}_\infty) \cong D(B, (B')^{coop}).$$

**Proof:** First of all, let us define a linear map:  $\phi: D(B, (B')^{coop}) \longrightarrow U_{r,s}(\mathfrak{sl}_\infty)$  as follows

$$\begin{aligned}\phi(\hat{\omega}_i^{\pm 1}) &= \omega_i^{\pm 1}, & \phi((\hat{\omega}'_i)^{\pm 1}) &= (\omega'_i)^{\pm 1} \\ \phi(\hat{e}_i) &= e_i, & \phi(\hat{f}_i) &= f_i.\end{aligned}$$

We need to show that this mapping is a Hopf algebra automorphism. Obviously, we can still employ the proof used in [5] for the finite case  $g = \mathfrak{sl}_{n+1}$  and we will not repeat the detail here.  $\square$

**1.2. An integral form of the two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$ .** In addition, we also consider an integral form of the two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$  and its subalgebras following [20]. Recall that we have set  $\mathcal{Z} = \mathbb{Z}[r^{\pm 1}, s^{\pm 1}]$ . For any  $l \geq 1$ , let us set the following

$$[l] = \frac{r^l - s^l}{r - s}, \quad [l]^! = [1][2] \cdots [l].$$

Let us define  $e_i^{(l)} = \frac{e_i^l}{[l]^!}, f_i^{(l)} = \frac{f_i^l}{[l]^!}$ . We define a  $\mathcal{Z}$ -subalgebra  $U_{r,s}(\mathfrak{sl}_\infty)_{\mathcal{Z}}$  of  $U_{r,s}(\mathfrak{sl}_\infty)$  which is generated by the elements  $e_i^{(l)}, f_i^{(l)}, w_i^{\pm 1}, w_i'^{\pm 1}$  for  $i \in I$ . Similarly, we can define the integral form of  $U_{r,s}^+(\mathfrak{sl}_\infty)$  and  $U_{r,s}^-(\mathfrak{sl}_\infty)$ . It is easy to see that we have the following

$$U_{r,s}(\mathfrak{sl}_\infty) \cong U_{r,s}(\mathfrak{sl}_\infty)_{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathbb{Q}(r, s)$$

and

$$U_{r,s}^\pm(\mathfrak{sl}_\infty) \cong U_{r,s}^\pm(\mathfrak{sl}_\infty)_{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathbb{Q}(r, s).$$

In particular,  $U_{r,s}(\mathfrak{sl}_\infty)$  (resp.  $U_{r,s}^\pm(\mathfrak{sl}_\infty)$ ) is a free  $\mathcal{Z}$ -algebra.

## 2. TWO-PARAMETER RINGEL-HALL ALGEBRAS $H_{r,s}(A_\infty)$

To better understand the structure of the two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$ , it is helpful to study its subalgebras  $U_{r,s}^+(\mathfrak{sl}_\infty)$  and (Hopf) subalgebra  $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$ . To this end, we shall study these algebras in terms of two-parameter Ringel-Hall algebras associated to the infinite linear quiver. In this section, we will define and study a two-parameter Ringel-Hall algebra  $H_{r,s}(A_\infty)$  associated to the category of finite dimensional representations of the infinite quiver

$$A_\infty: \bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet \xrightarrow{3} \cdots \bullet \xrightarrow{n-1} \bullet \xrightarrow{n} \cdots$$

For  $n \geq 1$ , let  $A_n$  denote the finite quiver

$$A_n: \bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet \xrightarrow{3} \cdots \bullet \xrightarrow{n-1} \bullet \xrightarrow{n} \bullet$$

with  $n$  vertices. Let us fix  $k$  to be a finite field and let  $\Lambda_n$  denote the path algebra of the finite linear quiver  $A_n$  over  $k$ . Then  $\Lambda_n$  is a finite



dimensional hereditary algebra of finite-representation type. Note that the category of finite dimensional representations of the quiver  $A_n$  is equivalent to the category of finite dimensional  $\Lambda_n$ -modules. We will denote this category by  $A_n\text{-mod}$ . Let us set  $q = |k|$  the cardinality of  $k$ , and choose  $v$  to be a number such that  $v^2 = q$ . We know that  $\Lambda_n$  is finitary in the sense that the cardinality of the extension group  $\text{Ext}^1(S, S')$  is finite for any two simple  $\Lambda_n$ -modules  $S, S'$ .

Let us denote by  $A_\infty\text{-mod}$ , the category of all finite dimensional representations of the quiver  $A_\infty$ . Note that the category  $A_\infty\text{-mod}$  has been investigated by Hou and Ye in [14], where they have explicitly described all finite dimensional indecomposable representations of  $A_\infty$  and studied the one-parameter non-twisted generic Ringel-Hall algebra  $H_q(A_\infty)$ . Let  $S_i$  be the simple representation associated to the vertex  $i$  of the quiver  $A_\infty$  and let  $M_{ij}$  denote the indecomposable representation of  $A_\infty$  with a top  $S_i$  and length  $j - i + 1$ . It is easy to see that there is a one-one correspondence between the set of isoclasses of finite dimensional indecomposable representations  $M_{ij}$  of the quiver  $A_\infty$  and the set of positive roots  $\alpha_{ij}$  for the Lie algebra  $\mathfrak{sl}_\infty$ .

Concerning the relationship between the categories  $A_n\text{-mod}$  and  $A_\infty\text{-mod}$ , we now recall the following result from [14].

**Theorem 2.1.** (*Theorem 1.1 in [14]*) *The category  $A_n\text{-mod}$  can be regarded as a fully faithful and extension closed subcategory of  $A_\infty\text{-mod}$  and  $A_m\text{-mod}$  for  $m \geq n$ .*

□

Based on the above theorem, we know that the extension group between any two finite dimensional representations  $M, N$  of  $A_\infty$  can be calculated via regarding  $M, N$  as the representations of a certain finite quiver  $A_m$ . Therefore, the number of extensions between  $M, N$  is still depicted by the evaluation of the Hall polynomial at  $q$ , the cardinality of the base field. Recall that the two-parameter Ringel-Hall algebra  $H_{r,s}(A_n)$ ,  $n \geq 1$  associated to the category  $A_n\text{-mod}$  has been studied in [24, 33]. In particular, one knows that  $H_{r,s}(A_n)$  can be presented as an iterated skew polynomial ring and its prime ideals are completely prime. A PBW basis has also been constructed for  $H_{r,s}(A_n)$  in [33] as well. In rest of this section, we are going to use a two-parameter twisted version of the Ringel-Hall algebra  $H_q(A_\infty)$  to study the algebra  $U_{r,s}^+(\mathfrak{sl}_\infty)$ . Note that this approach is plausible because of the existence of Hall polynomials in the category  $A_\infty\text{-mod}$ . Indeed, we will be looking at a limit version  $H_{r,s}(A_\infty)$  of the two-parameter Ringel-Hall algebras  $H_{r,s}(A_n)$ ,  $n \geq 1$ .

**2.1. Two-parameter Ringel-Hall algebra  $H_{r,s}(A_\infty)$ .** We will denote by  $\mathcal{P}$  the set of isomorphism classes of finite dimensional representations of the infinite quiver  $A_\infty$ . Let us define the subset

$$\mathcal{P}_1 = \mathcal{P} - 0$$

where 0 denotes the subset of  $\mathcal{P}$  consisting of the only isomorphism class of the zero representation. For any  $\alpha \in \mathcal{P}$ , we choose a representation  $u_\alpha$  corresponding to  $\alpha$ . We denote by  $a_\alpha$  the order of the automorphism group  $\text{Aut}(u_\alpha)$ . It is easy to see that the number  $a_\alpha$  is independent of the choices of the representatives  $u_\alpha$  for any  $\alpha \in \mathcal{P}$ .

For any given three representatives  $u_\alpha, u_\beta, u_\gamma$  of the elements  $\alpha, \beta, \gamma \in \mathcal{P}$  respectively, we denote by  $g_{\alpha\beta}^\gamma$  the number of submodules  $N$  of  $u_\gamma$  satisfying the conditions:  $N \cong u_\beta$  and  $u_\gamma/N \cong u_\alpha$ .

Note that it does not make sense to define  $\text{Ext}^1(M, N)$  for any two given representations  $M, N$  of the infinite quiver  $A_\infty$ . Let us denote by  $\hat{E}_{A_\infty}(M, N)$  the set of all short exact sequences  $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$ . We say two such short exact sequences  $0 \rightarrow N \rightarrow E_1 \rightarrow M \rightarrow 0$  and  $0 \rightarrow N \rightarrow E_2 \rightarrow M \rightarrow 0$  are equivalent if there exists a homomorphism  $\phi: E_1 \rightarrow E_2$  making the diagram commute. We denote by  $E_{A_\infty}(M, N)$  the set of all equivalence classes of  $\hat{E}_{A_\infty}$  with respect to this equivalence relation. For any given  $M, N \in A_\infty\text{-mod}$ , according to **Theorem 1.2** in [14], we can choose some  $m \geq 1$  such that there exists a bijection between  $E_{A_\infty}(M, N)$  and  $\text{Ext}_{A_m\text{-mod}}^1(M, N)$ . If no confusion arises, we will still write  $E_{A_\infty}(M, N)$  as  $\text{Ext}^1(M, N)$ .

For any given  $M, N \in A_\infty\text{-mod}$ , we define the following notation

$$\langle M, N \rangle = \dim_k \text{Hom}(M, N) - \dim_k \text{Ext}^1(M, N).$$

Once the representations  $M, N$  are chosen, we can always restrict to a subcategory  $A_n\text{-mod}$ . Since the algebra  $\Lambda_n$  is hereditary for any  $n \in \mathbb{N}$ , it is easy to see that for any representations  $M, N \in A_n\text{-mod}$ , the value of  $\langle M, N \rangle$  solely depends on the dimension vectors  $\underline{\dim} M, \underline{\dim} N$  of the  $A_n$ -modules  $M$  and  $N$ .

Now for any given elements  $\alpha, \beta \in \mathcal{P}$ , we can define the following notation

$$\langle \alpha, \beta \rangle = \langle u_\alpha, u_\beta \rangle$$

where  $u_\alpha, u_\beta$  are any chosen representatives of  $\alpha, \beta$  respectively. It is easy to see that  $\langle -, - \rangle$  is a bilinear form.

It is well known that in the category  $A_n\text{-mod}$ , there exists a symmetry between the objects of  $A_n\text{-mod}$ . This symmetry is described by Green's formula [15]. In fact, one can also prove that Green's formula holds for the objects in the category  $A_\infty\text{-mod}$ . Namely, we have the following result.

**Theorem 2.2.** *Let  $\alpha, \beta, \alpha', \beta' \in \mathcal{P}$ , then we have*

$$a_\alpha a_\beta a_{\alpha'} a_{\beta'} \sum_{\lambda \in \mathcal{P}} g_{\alpha, \beta}^\lambda g_{\alpha', \beta'}^\lambda a_\lambda^{-1} = \sum_{\rho, \sigma, \sigma', \tau \in \mathcal{P}} \frac{|Ext^1(u_\rho, u_\tau)|}{|Hom(u_\rho, u_\tau)|} g_{\rho\sigma}^\alpha g_{\rho\sigma'}^{\alpha'} g_{\sigma'\tau}^\beta g_{\sigma\tau}^{\beta'} a_\rho a_\sigma a_{\sigma'} a_{\tau'}.$$

**Proof:** Since all representations involved in the formula are finite dimensional representations of  $A_\infty$ , we can choose some positive integer  $m$  such that  $\alpha, \beta, \alpha', \beta'$  and  $\lambda$  can actually be regarded as objects in the subcategory  $A_m\text{-mod}$  instead. Note that Green's formula holds within the subcategory  $A_m\text{-mod}$ . Since the category  $A_m\text{-mod}$  is a fully faithful and extension closed subcategory of  $A_\infty\text{-mod}$ , we know that Green's formula holds in  $A_\infty\text{-mod}$ .  $\square$

Let  $H_{r,s}(A_n)$  denote the two-parameter Ringel-Hall algebra associated to the category  $A_n\text{-mod}$  as defined in [24]. In [24], Reineke has proved that the two-parameter Ringel-Hall algebra  $H_{r,s}(A_n)$  is isomorphic to the algebra  $U_{r,s}^+(\mathfrak{sl}n+1)$ . In the rest of this section, we will show that a limit version of this statement is still true.

Note that there exist Hall polynomials  $F_{M,N}^L(x)$  for  $M, N, L \in A_n\text{-mod}$  such that  $g_{M,N}^L = F_{M,N}^L(q)$ , where  $q$  is the cardinality of the base field  $k$ . For the existence and calculation of Hall polynomials in  $A_n\text{-mod}$ , we refer the reader to the references [27, 28]. Since each  $A_n\text{-mod}$  is a fully faithful and extension closed subcategory of  $A_\infty\text{-mod}$ , the Hall polynomials exists for objects in  $A_\infty\text{-mod}$ , which leads to the definition of two-parameter Ringel-Hall algebra  $H_{r,s}(A_\infty)$  below.

Now let us define  $H_{r,s}(A_\infty)$  to be the free  $\mathbb{Q}(r, s)$ -module generated by the set  $\{u_\alpha \mid \alpha \in \mathcal{P}\}$ . Moreover, we define a multiplication on the free  $\mathbb{Q}(r, s)$ -module  $H_{r,s}(A_\infty)$  as follows

$$u_\alpha u_\beta = \sum_{\lambda \in \mathcal{P}} s^{-(\alpha, \beta)} F_{u_\alpha u_\beta}^{u_\lambda}(rs^{-1}) u_\lambda, \quad \text{for any } \alpha, \beta \in \mathcal{P}.$$

It is easy to see that we have the following result.

**Theorem 2.3.** *The free  $\mathbb{Q}(r, s)$ -module  $H_{r,s}(A_\infty)$  is an associative  $\mathbb{Q}(r, s)$ -algebra under the above defined multiplication. In particular, the algebra  $H_{r,s}(A_n)$  can be regarded as a subalgebra of  $H_{r,s}(A_\infty)$  and  $H_{r,s}(A_m)$  for  $m \geq n$ . In particular, we have*

$$H_{r,s}(A_\infty) = \lim_{n \rightarrow \infty} H_{r,s}(A_n).$$

**Proof:** It is straightforward to verify that  $H_{r,s}(A_\infty)$  is an associative algebra under the above defined multiplication. Once again, we can reduce the proof to the finite case thanks to **Theorem 1.1** in [14]. Since each  $A_n\text{-mod}$  can be regarded as a fully faithful and extension closed subcategory of  $A_\infty\text{-mod}$  and  $A_m\text{-mod}$  when  $m \geq n$ ,

the algebra  $H_{r,s}(A_n)$  can be regarded as a subgroup of the algebras  $H_{r,s}(A_\infty)$  and  $H_{r,s}(A_m)$ . Furthermore, one notices that the multiplication of  $H_{r,s}(A_n)$  is the restriction of the multiplications of  $H_{r,s}(A_\infty)$  and  $H_{r,s}(A_m)$ . Therefore, the algebra  $H_{r,s}(A_n)$  can be regarded as a subalgebra of  $H_{r,s}(A_\infty)$  and  $H_{r,s}(A_m)$  for  $m \geq n$  as desired. Furthermore, each element of  $H_{r,s}(A_\infty)$  can be regarded as an element of a certain subalgebra  $H_{r,s}(A_m)$ . Thus we shall have  $H_{r,s}(A_\infty) = \lim_{n \rightarrow \infty} H_{r,s}(A_n)$  as desired.  $\square$

**2.2. Basic properties of  $H_{r,s}(A_\infty)$ .** Since the category  $A_\infty\text{-mod}$  can be regarded the direct limit of its fully faithful and extension closed subcategories  $A_n\text{-mod}$  with  $n \geq 1$ , any two objects  $M, N \in A_\infty\text{-mod}$  can be regarded as objects in a certain subcategory  $A_m\text{-mod}$ . Thus the extension between any such two objects can be handled in this subcategory  $A_n\text{-mod}$  as well. As a result, it is no surprise that the algebra  $H_{r,s}(A_\infty)$  shares many similar ring-theoretic properties with its subalgebras  $H_{r,s}(A_n)$ . In this subsection, we will establish some similar results for  $H_{r,s}(A_\infty)$  without giving detailed proofs. The reader shall be reminded that all the proofs can be reconstructed the same way as in the case of a certain subalgebra  $H_{r,s}(A_m)$ . And we refer the curious reader to [33] for the details.

First of all, let us fix more notations. For any given  $\alpha \in \mathcal{P}$ , we will choose an element  $u_\alpha \in H_{r,s}(\Lambda)$ . We denote by  $\epsilon(\alpha)$  the  $k$ -dimension of the endomorphism ring of the representative  $u_\alpha$  associated to  $\alpha$ . For any given finite dimensional representation  $M$  of the infinite quiver  $A_\infty$ , we will denote the isomorphism class of  $M$  by  $[M]$  and the dimension vector of  $M$  by  $\underline{\dim} M$ , which is an element of the Grothendieck group  $K_0(A_\infty)$  of the category  $A_\infty\text{-mod}$ .

Recall that there is a one-to-one correspondence between the set of all positive roots for the Lie algebra  $\mathfrak{sl}_\infty$  and the set of isoclasses of finite dimensional indecomposable representations of  $A_\infty$ . Let  $\mathbf{a} \in \Phi^+$  be any positive root, we shall denote by  $M(\mathbf{a})$  the indecomposable representation corresponding to  $\mathbf{a}$ . For any given map  $\alpha: \Phi^+ \rightarrow \mathbb{N}_0$  with finite support, let us set the following

$$M(\alpha) = M_\Lambda(\alpha) = \bigoplus_{\mathbf{a} \in \Phi^+} \alpha(\mathbf{a}) \mathbf{M}(\mathbf{a}).$$

Then it is easy to see there is a one-to-one correspondence between the set  $\mathcal{P}$  of isomorphism classes of all finite dimensional representations of the infinite quiver  $A_\infty$  and the set of all maps  $\alpha: \Phi^+ \rightarrow \mathbb{N}_0$  with finite supports. From now on, we will not distinguish an element  $\alpha \in \mathcal{P}$

from the corresponding map associated to  $\alpha$ , and we may denote both of them by  $\alpha$  if no confusion arises.

For any given  $\alpha \in \mathcal{P}$ , let us set  $\mathbf{dim}\alpha = \sum_{\mathbf{a} \in \Phi^+} \alpha(\mathbf{a})\mathbf{a}$ . Then we shall have following

$$\underline{dim}M(\alpha) = \mathbf{dim}\alpha.$$

For any given  $\alpha \in \mathcal{P}$ , we will denote by  $dim(\alpha) = dim(u_\alpha)$  the dimension of the representation  $u_\alpha$  as a  $k$ -vector space. Furthermore, let us set

$$\langle u_\alpha \rangle = s^{dim(u_\alpha) - \epsilon(\alpha)} u_\alpha.$$

For conveniences, we may sometimes simply denote the element  $u_\alpha$  by  $\alpha$  for any  $\alpha \in \mathcal{P}$ , and denote  $F_{u_\alpha u_\beta}^{u_\lambda}(rs^{-1})$  by  $g_{\alpha\beta}^\lambda$ , if no confusion arises. In the rest of this subsection, we will carry out all the computations in terms of  $\alpha$ . It is obvious that the set  $\{\langle \alpha \rangle \mid \alpha \in \mathcal{P}\}$  is also a  $\mathbb{Q}(r, s)$ -basis for the algebra  $H_{r,s}(A_\infty)$ . Note that we have  $\langle \alpha_i \rangle = \alpha_i$  for any given element  $\alpha_i \in \mathcal{P}$  corresponding to the simple root  $\alpha_i, i \geq 1$ . As a result, we can rewrite the multiplication of  $H_{r,s}(A_\infty)$  in terms of this new basis as follows

$$\langle \alpha \rangle \langle \beta \rangle = s^{-\epsilon(\alpha) - \epsilon(\beta) - \langle \mathbf{dim}\alpha, \mathbf{dim}\beta \rangle} \sum_{\lambda \in \mathcal{P}} s^{\epsilon(\lambda)} g_{\alpha\beta}^\lambda \langle \lambda \rangle$$

for any  $\alpha, \beta \in \mathcal{P}$ .

In addition, let us denote by

$$e(\alpha, \beta) = dim_k Hom_{A_\infty-mod}(M(\alpha), M(\beta))$$

and

$$\zeta(\alpha, \beta) = dim_k Ext_{A_\infty-mod}^1(M(\alpha), M(\beta)).$$

Recall that Hou and Ye have given an explicit total ordering on the set of all isoclasses of finite dimensional indecomposable representations of the infinite linear quiver  $A_\infty$  and used it to construct a PBW base for the generic one-parameter Ringel-Hall algebra  $H_q(A_\infty)$ . Following [14], we will order all the positive roots as follows:

$$\mathbf{a}_{11} < \mathbf{a}_{12} < \cdots < \mathbf{a}_{22} < \mathbf{a}_{23} < \cdots.$$

Obviously, we can see that  $Hom(M(\mathbf{a}_{ij}), M(\mathbf{a}_{kl})) \neq 0$  implies  $\mathbf{a}_{ij} > \mathbf{a}_{kl}$ , where  $M(\mathbf{a}_{ij}), M(\mathbf{a}_{kl})$  are the indecomposable representations corresponding to the positive roots  $\mathbf{a}_{ij}, \mathbf{a}_{kl}$  respectively. For more details about the ordering, we refer the reader to [14, 27]. We should mention that we may write the positive roots as  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots$  instead.

First of all, we have the following proposition.

**Proposition 2.1.** *Let  $\alpha_1, \dots, \alpha_t \in \mathcal{P}$  such that for  $i < j$ , we have both  $\epsilon(\alpha_j, \alpha_i) = 0$  and  $\zeta(\alpha_i, \alpha_j) = 0$ . Then*

$$\langle \bigoplus_{i=1}^t \alpha_i \rangle = \langle \alpha_1 \rangle \cdots \langle \alpha_t \rangle.$$

□

**Theorem 2.4.** *Let  $\alpha, \beta \in \mathcal{P}$  such that  $e(\beta, \alpha) = 0, \zeta(\alpha, \beta) = 0$ . Then we have the following*

$$\langle \beta \rangle \langle \alpha \rangle = r^{\langle \alpha, \beta \rangle} s^{-\langle \beta, \alpha \rangle} \langle \alpha \rangle \langle \beta \rangle + \sum_{\gamma \in J(\alpha, \beta)} c_\gamma \langle \gamma \rangle$$

with coefficients  $c_\gamma$  in  $\mathbb{Z}[r^{\pm 1}, s^{\pm 1}]$  and  $J(\alpha, \beta)$  is the set of all elements  $\lambda \in \mathcal{P}$  which are different from  $\alpha \oplus \beta$  and  $g_{\alpha\beta}^\lambda \neq 0$ .

□

**Proposition 2.2.** *For any given  $\alpha \in \mathcal{P}$ , we have*

$$\langle \alpha \rangle = \langle \alpha(\mathbf{a}_1)\mathbf{a}_1 \rangle \cdots \langle \alpha(\mathbf{a}_m)\mathbf{a}_m \rangle.$$

□

Now let us consider the divided powers of  $\langle \mathbf{a} \rangle$  by setting

$$\langle \mathbf{a} \rangle^{(t)} = \frac{1}{[t]_{\epsilon(\mathbf{a})}!} \langle \mathbf{a} \rangle^t$$

where  $[t]_{\epsilon(\mathbf{a})}! = \prod_{i=1}^t \frac{r^{i\epsilon(\mathbf{a})} - s^{i\epsilon(\mathbf{a})}}{r^{\epsilon(\mathbf{a})} - s^{\epsilon(\mathbf{a})}}$ .

Then we have the following lemma.

**Lemma 2.1.** *Let  $\mathbf{a}$  be a positive root and  $t \geq 0$  be an integer. Then we have the following*

$$\langle t\mathbf{a} \rangle = \langle \mathbf{a} \rangle^{(t)}.$$

□

For each positive root  $\mathbf{a}_i$ , let us define the following symbol

$$X_i = \langle \mathbf{a}_i \rangle.$$

Then we have the following proposition:

**Proposition 2.3.** *Let  $\alpha \in \mathcal{P}$  and regard  $\alpha$  as a map  $\alpha: \Phi^+ \rightarrow \mathbb{N}_0$  with finite support. Let us set  $\alpha(i) = \alpha(\mathbf{a}_i)$ , then we have the following*

$$\langle \alpha \rangle = X_1^{\alpha(1)} \cdots X_m^{\alpha(m)} = \left( \prod_{i=1}^m \frac{1}{[\alpha(i)]_{\epsilon(\mathbf{a}_i)}!} \right) X_1^{\alpha(1)} \cdots X_m^{\alpha(m)}.$$

□

**Theorem 2.5.** *The monomials  $X_1^{\alpha(1)} \cdots X_m^{\alpha(m)}$  with  $\alpha(1), \dots, \alpha(m) \in \mathbb{N}_0$  form a  $\mathbb{Q}(r, s)$ -basis of  $H_{r,s}(\Lambda)$ ; and for  $i < j$ , we have*

$$\begin{aligned} X_j X_i &= r^{\langle \underline{\dim} X_i, \underline{\dim} X_j \rangle} s^{-\langle \underline{\dim} X_j, \underline{\dim} X_i \rangle} X_i X_j \\ &\quad + \sum_{I(i,j)} c(a_{i+1}, \dots, a_{j-1}) X_{i+1}^{a_{i+1}} \cdots X_{j-1}^{a_{j-1}} \end{aligned}$$

with coefficients  $c(a_{i+1}, \dots, a_{j-1})$  in  $\mathbb{Q}(r, s)$ . Here the index set  $I(i, j)$  is the set of sequences  $(a_{i+1}, \dots, a_{j-1})$  of natural numbers such that  $\sum_{t=i+1}^{j-1} a_t \mathbf{a}_t = \mathbf{a}_i + \mathbf{a}_j$ .

□

Now we define some algebra automorphisms and skew derivations on  $H_{r,s}(A_\infty)$ . For any  $d \in \mathbb{Z}^{\oplus \mathbb{N}}$ , we define an algebra automorphism  $l_d$  of  $H_{r,s}(A_\infty)$  as follows

$$l_d(w) = r^{\langle \underline{\dim} w, d \rangle} s^{-\langle d, \underline{\dim} w \rangle} w$$

where  $w$  is any homogeneous element of  $H_{r,s}(A_\infty)$ .

Let  $H_j$  denote the  $\mathbb{Q}(r, s)$ -subalgebra of  $H_{r,s}(A_\infty)$  generated by the generators  $X_1, \dots, X_j$ . Thus we have  $H_0 = \mathbb{Q}(r, s)$  and for any  $0 \leq j \leq m$ , we have following

$$H_j = H_{j-1}[X_j, l_j, \delta_j]$$

with the automorphism  $l_j$  and the  $l_j$ -derivation  $\delta_j$  of  $H_{j-1}$ . Note that the automorphism  $l_j$  can be explicitly defined as follows

$$l_j(X_i) = r^{\langle \underline{\dim} X_i, \underline{\dim} X_j \rangle} s^{-\langle \underline{\dim} X_j, \underline{\dim} X_i \rangle} X_i$$

for  $i < j$ . And the skew derivation  $\delta_j$  can be defined as follows:

$$\delta_j(X_i) = X_j X_i - l_j(X_i) X_j = \sum_{I(i,j)} c(a_{i+1}, \dots, a_{j-1}) X_{i+1}^{a_{i+1}} \cdots X_{j-1}^{a_{j-1}}.$$

It is easy to check that we have the following result.

**Proposition 2.4.** *The automorphism  $l_j$  and the skew derivation  $\delta_j$  satisfy the following relation*

$$l_j \delta_j = r^{\langle \mathbf{a}_j, \mathbf{a}_j \rangle} s^{-\langle \mathbf{a}_j, \mathbf{a}_j \rangle} \delta_j l_j.$$

□

**Theorem 2.6.** *The two-parameter Ringel-Hall algebra  $H_{r,s}(A_\infty)$  can be presented as an iterated skew polynomial ring.*

□

Let  $R$  be a ring. Recall that a two-sided ideal  $P \subset R$  is said to be prime if  $P \neq R$  and whenever the product  $AB$  of two two-sided ideals

$A, B$  of  $R$  is contained in  $P$ , then at least one of  $A$  and  $B$  is contained in  $P$ . A two-sided ideal  $P \subset R$  is called completely prime if  $P \neq R$  and whenever the product  $ab$  of two elements of  $R$  is contained in  $P$ , then at least one of the elements  $a$  and  $b$  is contained in  $P$ . In the case of commutative rings, prime ideals are exactly completely prime ideals. In the case of noncommutative rings, a completely prime ideal is a prime ideal; but a prime ideal may not necessarily be a completely prime ideal. We refer the reader to [2] for more details. Concerning prime ideals of the algebra  $H_{r,s}(A_\infty)$ , we have the following result.

**Corollary 2.1.** *Suppose the multiplicative group generated by  $r, s$  is torsion-free, then any prime ideal of  $H_{r,s}(A_\infty)$  is completely prime.*

**Proof:** Let  $P \subset H_{r,s}(A_\infty)$  be a prime ideal of  $H_{r,s}(A_\infty)$ . Since  $H_{r,s}(A_\infty) = \lim_{n \rightarrow \infty} H_{r,s}(A_n)$ , we have that  $P \cap H_{r,s}(A_n) \subset H_{r,s}(A_n)$  is a prime ideal of  $H_{r,s}(A_n)$  for any  $n \geq 1$ . Let  $a, b \in H_{r,s}(A_\infty)$  such that  $ab \in P$ . Then we can choose  $m \in \mathbb{N}$  such that  $a, b, ab \in H_{r,s}(A_m)$ . By the result in [33], we know that all prime ideals of  $H_{r,s}(A_m)$ , ( $m \geq 1$ ) are completely prime. Therefore, we have that  $a \in H_{r,s}(A_m)$  or  $b \in H_{r,s}(A_m)$ . Hence, the prime ideal  $P$  is a completely prime ideal of  $H_{r,s}(A_\infty)$ .  $\square$

**2.3. An algebra isomorphism from  $U_{r,s}^+(\mathfrak{sl}_\infty)$  onto  $H_{r,s}(A_\infty)$ .** In this subsection, we are going to establish a graded algebra isomorphism from the two-parameter quantized enveloping algebra  $U_{r,s}^+(\mathfrak{sl}_\infty)$  onto the two-parameter Ringel-Hall algebra  $H_{r,s}(A_\infty)$ . Via this isomorphism, all results established in the previous subsection on  $H_{r,s}(A_\infty)$  can be transformed to the two-parameter quantized enveloping algebra  $U_{r,s}^+(\mathfrak{sl}_\infty)$ . Indeed, the isomorphism from  $U_{r,s}^+(\mathfrak{sl}_\infty)$  onto  $H_{r,s}(A_\infty)$  is the direct limit of the isomorphisms from  $U_{r,s}^+(\mathfrak{sl}_{n+1})$  onto  $H_{r,s}(A_n)$ .

First of all, one can prove the following result, which induces a homomorphism from  $U_{r,s}^+(\mathfrak{sl}_\infty)$  into  $H_{r,s}(A_\infty)$ .

**Lemma 2.2.** *Let  $\alpha_i \in \mathcal{P}$  correspond to the simple module  $S_i$ , then we have the following identities in  $H_{r,s}(A_\infty)$ .*

$$\begin{aligned} \alpha_i^2 \alpha_{i+1}^2 - (r+s) \alpha_i \alpha_{i+1} \alpha_i + r s \alpha_{i+1} \alpha_i^2 &= 0, \\ \alpha_i \alpha_{i+1}^2 - (r+s) \alpha_{i+1} \alpha_i \alpha_{i+1} + r s \alpha_i \alpha_{i+1}^2 &= 0, \end{aligned}$$

for  $i = 1, 2, 3, \dots$ .

**Proof:** Note that we can regard  $\alpha_i, \alpha_{i+1}$  as elements of the two-parameter Ringel-Hall algebra  $H_{r,s}(A_{i+1})$ , which is a subalgebra of  $H_{r,s}(A_\infty)$ . By the result in [33], we know that these identities hold



in the algebra  $H_{r,s}(A_{i+1})$ . Therefore, we have proved the result as desired.  $\square$

Now we have the following result which relates Ringel-Hall  $H_{r,s}(A_\infty)$  to the algebra  $U_{r,s}^+(\mathfrak{sl}_\infty)$ .

**Theorem 2.7.** *The map*

$$\eta: e_i \longrightarrow \alpha_i$$

*extends to a  $\mathbb{Q}(r, s)$ -algebra isomorphism*

$$\eta: U_{r,s}^+(\mathfrak{sl}_\infty) \longrightarrow H_{r,s}(A_\infty).$$

**Proof:** (The proof is essentially borrowed from [24] and we include it for completeness. See also [33]). First of all, note that the quantum Serre relations of  $U_{r,s}^+(\mathfrak{sl}_\infty)$  are preserved by the map  $\eta$ . Thus the map  $\eta$  does defines an algebra homomorphism from the two-parameter quantized enveloping algebra  $U_{r,s}^+(\mathfrak{sl}_\infty)$  into the two-parameter twisted Ringel-Hall algebra  $H_{r,s}(A_\infty)$ . Now it suffices to show that the map  $\eta$  is indeed a bijection.

We first show that the map  $\eta$  is surjective by verifying that the algebra  $H_{r,s}(A_\infty)$  is generated by the elements  $u_i$  which correspond to the irreducible representation  $S_i$  of the infinite quiver  $A_\infty$ . Let  $u_\alpha$  be any element in  $H_{r,s}(A_\infty)$ , then we can regard  $u_\alpha$  as an element of a certain subalgebra  $H_{r,s}(A_n)$ . Thus we can restrict our proof to the subalgebra  $H_{r,s}(A_n)$ . As a result, we have the following:

$$u_\alpha = \left( \prod_{i=1}^m \frac{1}{[\alpha(i)]_{\epsilon(\mathbf{a}_i)}!} \right) u_{\mathbf{a}_1}^{\alpha(\mathbf{a}_1)} \cdots u_{\mathbf{a}_m}^{\alpha(\mathbf{a}_m)}.$$

Now we need to prove that  $u_\alpha$  is generated by  $u_i$  for any  $\alpha$  corresponding to an indecomposable representations. We prove this claim by using induction. Note that  $\zeta(\alpha, \alpha) = 0$ , thus we have the following

$$u_\alpha = u_1^{d_1} \cdots u_n^{d_n} - \sum_{\beta \neq \alpha, \dim \beta = \dim \alpha} s^{(\beta, \beta)} u_\beta.$$

However, one sees that the dimension of the module  $u_\beta$  is less than the dimension of the module  $u_\alpha$ . Thus by induction on the dimension, we can reduce to the case where  $\dim(u_\alpha) = 1$ . In this case, the only possibility is that  $u_\alpha = u_i$  for some  $i$ . Thus we have proved the statement that every  $u_\alpha$  is generated by  $u_i$ , which further implies that the map  $\eta$  is a surjective map. We also note that the map  $\eta$  is a graded map.

Finally, we show that the map  $\eta$  is also injective. Recall that  $\mathcal{A} = \mathbb{Q}[r, s]_{(r-1, s-1)}$  denote the localization of the polynomial ring  $\mathbb{Q}[r, s]$  at

the maximal ideal  $(r-1, s-1)$ . Then we know that  $\mathcal{A} = \mathbb{Q}[r, s]_{(r-1, s-1)}$  is a local ring with the residue field  $\mathbb{Q}$  and the fractional field  $\mathbb{Q}(r, s)$ . Let  $U_{\mathcal{A}}^+$  denote the free  $\mathcal{A}$ -algebra generated by the generators  $e_i$  subject to the quantum Serre relations holding in  $U_{r,s}^+(\mathfrak{sl}_{\infty})$ . Also let  $U_{\mathbb{Q}}^+(\mathfrak{sl}_{\infty})$  denote the universal enveloping algebra of the corresponding nilpotent Lie subalgebra  $\mathfrak{n}^+$  of  $\mathfrak{sl}_{\infty}$  defined over the base field  $\mathbb{Q}$ . Then we have the following

$$U_{r,s}^+(\mathfrak{sl}_{\infty}) = \mathbb{Q}(r, s) \otimes_{\mathcal{A}} U_{\mathcal{A}}^+, \quad U_{\mathbb{Q}}^+(\mathfrak{sl}_{\infty}) = \mathbb{Q} \otimes_{\mathcal{A}} U_{\mathcal{A}}^+.$$

For any  $\beta \in \mathbb{Z}^{\oplus \mathbb{N}}$ , we have the following result via Nakayama's Lemma

$$\begin{aligned} \dim_{\mathbb{Q}} U_{\mathbb{Q}}^+(\mathfrak{sl}_{\infty})_{\beta} &= \dim_{\mathbb{Q}} (\mathbb{Q} \otimes_{\mathcal{A}} U_{\mathcal{A}}^+)_{\beta} \\ &\geq \dim_{\mathbb{Q}(r,s)} (\mathbb{Q}(r, s) \otimes_{\mathcal{A}} U_{\mathcal{A}}^+)_{\beta} \\ &= \dim_{\mathbb{Q}(r,s)} U_{r,s}^+(\mathfrak{sl}_{\infty})_{\beta} \\ &\geq \dim_{\mathbb{Q}(r,s)} H_{r,s}(A_{\infty})_{\beta}. \end{aligned}$$

Using **Corollary 2** in the reference [28] and the PBW-theorem, we also have the following result:

$$\dim_{\mathbb{Q}} U_{\mathbb{Q}}^+(\mathfrak{sl}_{\infty})_{\beta} = \dim_{\mathbb{Q}(r,s)} H_{r,s}(A_{\infty})_{\beta}.$$

Thus we have proved that the map  $\eta$  is injective. Therefore, the map  $\eta$  is an algebra isomorphism from  $U_{r,s}^+(\mathfrak{sl}_{\infty})$  onto  $H_{r,s}(A_{\infty})$  as desired.  $\square$

$\square$

Based on the previous theorem, the following corollary is in order.

**Corollary 2.2.** *The algebra  $U_{r,s}^+(\mathfrak{sl}_{\infty})$  has a  $\mathbb{Q}(r, s)$ -basis parameterized by the isomorphism classes of all finite dimensional representations of the infinite quiver  $A_{\infty}$ . In particular, we have*

$$U_{r,s}^+(\mathfrak{sl}_{\infty}) = \lim_{n \rightarrow \infty} U_{r,s}^+(\mathfrak{sl}_{n+1}).$$

$\square$

**Corollary 2.3.** *All prime ideals of  $U_{r,s}^+(\mathfrak{sl}_{\infty})$  are completely prime under the condition that the multiplicative group generated by  $r, s$  is torsion-free.*

$\square$

### 3. THE EXTENDED TWO-PARAMETER RINGEL-HALL ALGEBRAS $\overline{H_{r,s}(A_\infty)}$

For the purpose of realizing the Borel subalgebra  $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$  of the two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$ , we define the extended Ringel-Hall algebra  $\overline{H_{r,s}(A_\infty)}$  by adding the torus part. In particular, we show that there is a Hopf algebra structure on this extended two-parameter Ringel-Hall algebra  $\overline{H_{r,s}(A_\infty)}$ ; as a result we prove that  $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$  is isomorphic to the extended two-parameter Ringel-Hall algebra  $\overline{H_{r,s}(A_\infty)}$  as a Hopf algebra. Similarly, we can use an extended two-parameter Ringel-Hall algebra to realize the Borel subalgebra  $U^{\leq 0}(\mathfrak{sl}_\infty)$ . Therefore, we will obtain a PBW-basis of two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$ .

**3.1. Extended Ringel-Hall algebras  $\overline{H_{r,s}(A_\infty)}$ .** Let us define  $\overline{H_{r,s}(A_\infty)}$  to be a free  $\mathbb{Q}(r, s)$ -module with the following basis

$$\{k_\alpha u_\lambda \mid \alpha \in \mathbb{Z}[I], \lambda \in \mathcal{P}\}.$$

Moreover one will define an algebra structure on the module  $\overline{H_{r,s}(A_\infty)}$  as follows.

$$\begin{aligned} u_\alpha u_\beta &= \sum_{\lambda \in \mathcal{P}} s^{-\langle \alpha, \beta \rangle} F_{u_\alpha, u_\beta}^{u_\lambda} (rs^{-1}) u_\lambda, \quad \text{for any } \alpha, \beta \in \mathcal{P}, \\ k_\alpha u_\beta &= r^{\langle \beta, \alpha \rangle} s^{-\langle \alpha, \beta \rangle} u_\beta k_\alpha \quad \text{for any } \alpha \in \mathbb{Z}[I], \beta \in \mathcal{P}, \\ k_\alpha k_\beta &= k_\beta k_\alpha \quad \text{for any } \alpha, \beta \in \mathbb{Z}[I]. \end{aligned}$$

Indeed, we have the following

**Proposition 3.1.** *For any elements  $x, y, z \in \mathbb{Z}[I]$  and  $\alpha, \beta, \gamma \in \mathcal{P}$ , we have the following*

$$[(k_x u_\alpha)(k_y u_\beta)](k_z u_\gamma) = (k_x u_\alpha)[(k_y u_\beta)(k_z u_\gamma)].$$

*In particular, with the above defined multiplication,  $\overline{H_{r,s}(A_\infty)}$  is an associative  $\mathbb{Q}(r, s)$ -algebra.*

**Proof:** Once we choose  $x, y, z$ , and  $\alpha, \beta, \gamma$ , we can restrict to the subgroup  $\overline{H_{r,s}(A_m)}$  of  $\overline{H_{r,s}(A_\infty)}$  for some  $m$ . Since  $\overline{H_{r,s}(A_m)}$  is an associative algebra with the restricted multiplication, thus we have proved all the statements.  $\square$

Furthermore, we have the following result.

**Theorem 3.1.** *The map  $\eta$  extends to a  $\mathbb{Q}(r, s)$ -algebra isomorphism from  $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$  onto  $\overline{H_{r,s}(A_\infty)}$  via the map  $\eta(w_i) = k_i$  and  $\eta(e_i) = u_{\alpha_i}$ .*

**Proof:** The proof is straightforward.  $\square$

As a result, we have the following description about a basis for the algebra  $U_{r,s}^+(\mathfrak{sl}_\infty)$

**Corollary 3.1.** *The set  $\mathbf{B}^+ = \{w_\alpha \eta^{-1}(u_\lambda) \mid \alpha \in \mathbb{Z}[\mathbb{N}], \lambda \in \mathcal{P}\}$  is a  $\mathbb{Q}(r, s)$ -basis of  $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$ .*  $\square$

**3.2. A Hopf algebra structure on  $\overline{H_{r,s}(A_\infty)}$ .** Now we are going to introduce a Hopf algebra structure on the extended two-parameter Ringel-Hall algebra  $\overline{H_{r,s}(A_\infty)}$ . In particular, we have the following result.

**Theorem 3.2.** *The algebra  $\overline{H_{r,s}(A_\infty)}$  is a Hopf algebra with the Hopf algebra structure defined as follows.*

(1) *Multiplication:*

$$\begin{aligned} u_\alpha u_\beta &= \sum_{\lambda \in \mathcal{P}} s^{-\langle \alpha, \beta \rangle} g_{\alpha\beta}^\lambda u_\lambda \quad \text{for any } \alpha, \beta \in \mathcal{B}, \\ k_\alpha u_\beta &= r^{\langle \beta, \alpha \rangle} s^{-\langle \alpha, \beta \rangle} u_\beta k_\alpha \quad \text{for any } \alpha \in \mathbb{Z}[I], \beta \in \mathcal{P}, \\ k_\alpha k_\beta &= k_\beta k_\alpha \quad \text{for any } \alpha, \beta \in \mathbb{Z}[I]. \end{aligned}$$

(2) *Comultiplication:*

$$\begin{aligned} \Delta(u_\lambda) &= \sum_{\alpha, \beta \in \mathcal{P}} r^{\langle \alpha, \beta \rangle} \frac{a_\alpha a_\beta}{a_\lambda} g_{\alpha\beta}^\lambda u_\alpha k_\beta \otimes u_\beta \quad \text{for any } \lambda \in \mathcal{P}, \\ \Delta(k_\alpha) &= k_\alpha \otimes k_\alpha \quad \text{for any } \alpha \in \mathbb{Z}[I]. \end{aligned}$$

(3) *Counit:*

$$\epsilon(u_\lambda) = 0 \quad \text{for all } \lambda \neq 0 \quad \text{and} \quad \epsilon(k_\alpha) = 1 \quad \text{for any } \alpha \in \mathcal{P}.$$

(4) *Antipode:*

$$\begin{aligned} \sigma(u_\lambda) &= \delta_{\lambda,0} + \sum_{m \geq 1} (-1)^m \times \sum_{\pi \in \mathcal{P}, \lambda_1, \lambda_2, \dots, \lambda_m \in \mathcal{P}_1} (rs^{-1})^{\sum_{i < j} \langle \lambda_i, \lambda_j \rangle} \\ &\quad \frac{a_{\lambda_1} \cdots a_{\lambda_m}}{a_\lambda} g_{\lambda_1 \dots \lambda_m}^\lambda g_{\lambda_1 \dots \lambda_m}^\pi k_{-\lambda} u_\pi \\ &\quad \text{for any element } \lambda \in \mathcal{P} \text{ and} \end{aligned}$$

$$\sigma(k_\alpha) = k_{-\alpha} \quad \text{for any } \alpha \in \mathbb{Z}[I].$$

In particular, we have the following

$$\overline{H_{r,s}(A_\infty)} = \lim_{n \rightarrow \infty} \overline{H_{r,s}(A_n)}$$

as a direct limit of Hopf subalgebras.

□

The proof of the above theorem consists of a couple of lemmas which can be proved as the finite dimensional case. And we refer the reader to [33, 34] for more details. Namely, we have the following lemmas.

**Lemma 3.1.** *The comultiplication  $\Delta$  is an algebra endomorphism of  $\overline{H_{r,s}(A_\infty)}$ .*

□

**Lemma 3.2.** *For any  $\lambda \in \mathcal{P}$ , we have the following*

$$\mu(\sigma \otimes 1)\Delta(u_\lambda) = \delta_{\lambda 0}$$

and

$$\mu(1 \otimes \sigma)\Delta(u_\lambda) = \delta_{\lambda 0}.$$

□

**3.3. A Hopf algebra isomorphism from  $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$  onto  $\overline{H_{r,s}(A_\infty)}$ .**

In this subsection, we will prove that the Borel subalgebras  $U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty)$  and  $U_{r,s}^{\leq 0}(\mathfrak{sl}_\infty)$  of the two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$  can be realized as the extended two-parameter Ringel-Hall algebra  $\overline{H_{r,s}(A_\infty)}$  and  $\overline{H_{s^{-1},r^{-1}}(A_\infty)}$  as Hopf algebras. As a result, we shall derive a PBW-basis for the two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$ .

**Theorem 3.3.** *We have that*

$$U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty) \cong \overline{H_{r,s}(A_\infty)}$$

and

$$U_{r,s}^{\leq 0}(\mathfrak{sl}_\infty) \cong \overline{H_{s^{-1},r^{-1}}(A_\infty)}$$

as Hopf algebras.

□

Let  $\mathbf{B}^-$  denote the  $\mathbb{Q}(r, s)$ -basis constructed for the algebra  $U_{r,s}^{\leq 0}(\mathfrak{sl}_\infty)$  via the Ringel-Hall algebra  $\overline{H_{s^{-1},r^{-1}}(A_\infty)}$ , then we have the following:

**Corollary 3.2.** *The set  $\mathbf{B}^+ \times \mathbf{B}^-$  is a  $\mathbb{Q}(r, s)$ -basis for the two-parameter quantum groups  $U_{r,s}(\mathfrak{sl}_\infty)$ .*

□

Furthermore, we have the following result, which provides a bridge from the finite dimensional case to the infinite case.

**Theorem 3.4.** *The two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_\infty)$  is the direct limit of the direct system  $\{U_{r,s}(\mathfrak{sl}_{n+1}) \mid n \in \mathbb{N}\}$  of the Hopf subalgebras  $U_{r,s}(\mathfrak{sl}_{n+1})$  of  $U_{r,s}(\mathfrak{sl}_\infty)$ . That is*

$$U_{r,s}(\mathfrak{sl}_\infty) = \lim_{n \rightarrow \infty} U_{r,s}(\mathfrak{sl}_{n+1}).$$

*In particular, we have*

$$\begin{aligned} U_{r,s}^{\pm 1}(\mathfrak{sl}_\infty) &= \lim_{n \rightarrow \infty} U_{r,s}^{\pm 1}(\mathfrak{sl}_{n+1}), \\ U_{r,s}^0(\mathfrak{sl}_\infty) &= \lim_{n \rightarrow \infty} U_{r,s}^0(\mathfrak{sl}_{n+1}), \\ U_{r,s}^{\geq 0}(\mathfrak{sl}_\infty) &= \lim_{n \rightarrow \infty} U_{r,s}^{\geq 0}(\mathfrak{sl}_{n+1}), \\ U_{r,s}^{\leq 0}(\mathfrak{sl}_\infty) &= \lim_{n \rightarrow \infty} U_{r,s}^{\leq 0}(\mathfrak{sl}_{n+1}). \end{aligned}$$

**Proof:** It is obvious that  $U_{r,s}(\mathfrak{sl}_{n+1})$  are Hopf subalgebras of  $U_{r,s}(\mathfrak{sl}_\infty)$  and  $U_{r,s}(\mathfrak{sl}_{m+1})$  for  $m \geq n$ . In addition, any element of  $U_{r,s}(\mathfrak{sl}_\infty)$  is an element of a certain  $U_{r,s}(\mathfrak{sl}_{n+1})$ . Thus we are done with the proof.  $\square$

#### 4. MONOMIAL BASES AND BAR-INVARIANT BASES OF $U_{r,s}^+(\mathfrak{sl}_\infty)$

In this section, we study various bases of  $U_{r,s}^+(\mathfrak{sl}_\infty)$  via the theory of generic extensions. Note that the construction of monomial bases using generic extension theory for the Ringel-Hall algebras of type  $A, D, E$  has been done in [9]. The idea of the construction is to use the monoidal structure on the set  $\mathcal{M}$  of isoclasses of finite dimensional representations of the corresponding quiver  $\mathcal{Q}$  and the Bruhat-Chevalley type partial ordering in  $\mathcal{M}$ . Note that the arguments used in [9] can be completely transformed to the case of  $\mathfrak{sl}_\infty$ . Therefore, we will state most of the results for monomial bases without much detail. For the details, we refer the reader to [9, 24].

For the reader's convenience, we will recall the necessary details about the generic extensions from [9, 24]. Note that there exists a bijective correspondence between the set of positive roots  $\Phi^+$  of the root system  $\Phi$  associated to  $\mathfrak{sl}_\infty$  and the set of isoclasses of finite dimensional indecomposable representations of  $A_\infty$ . For any  $\beta \in \Phi^+$ , we will denote by  $M(\beta) = M_k(\beta)$  the corresponding indecomposable representation of  $A_\infty$ . By the Krull-Remak-Schmidt theorem, we shall have the following

$$M(\lambda) = M_k(\lambda) = \bigoplus_{\beta \in \Phi^+} \lambda(\beta) M_k(\beta)$$

for some function  $\lambda: \Phi^+ \rightarrow \mathbb{N}_0$  with a finite support. Therefore, the isoclasses of finite dimensional representations of  $A_\infty$  are indexed by

the following set

$$\Lambda = \{\lambda: \Phi^+ \longrightarrow \mathbb{N} \text{ with a finite support}\} \cong \mathbb{N}_0^{\oplus \Phi^+}.$$

From now on, we will use the set  $\Lambda$  to index the objects of the category  $A_\infty\text{-mod}$ .

Next, we are going to recall some information about generic extensions of representations of Dynkin quivers. We should mention that all the arguments used in the finite dimensional cases of type  $A, D, E$  can be transformed to the  $\mathfrak{sl}_\infty$ . We refer the interested reader to the references [9, 24] for details.

Let us fix  $k$  to algebraically closed. Let us denote by  $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$  the quiver  $A_\infty$ . Fix a  $\mathbf{d} = (d_i)_i \in \mathbb{N}_0^{\oplus \Phi^+}$  and we may choose  $n$  large enough so that  $\mathbf{d}$  can be regarded as an element in  $\mathbb{N}_0^n$ . For any given  $\mathbf{d}$ , we can define an affine space as follows

$$R(\mathbf{d}) = R(\mathcal{Q}, \mathbf{d}): = \prod_{\alpha \in \mathcal{Q}_1} \text{Hom}_k(k^{d_{t(\alpha)}}, k^{d_{h(\alpha)}}) \cong \prod_{\alpha \in \mathcal{Q}_1} k^{d_{t(\alpha)} \times d_{h(\alpha)}}.$$

Thus, a point  $x = (x_\alpha)_\alpha$  of  $R(\mathbf{d})$  determines a finite dimensional representation  $V(x)$  of  $\mathcal{Q} = A_\infty$ . The algebraic group  $GL(\mathbf{d}) = \prod_{i=1}^n GL_{d_i}(k)$  acts on the space  $R(\mathbf{d})$  by the conjugation

$$(g_i)_i \dots (x_\alpha)_\alpha = (g_{h(\alpha)})x_\alpha g_{t(\alpha)}^{-1})_\alpha.$$

and the  $GL(\mathbf{d})$ -orbits  $\mathcal{O}_x$  in  $R(\mathbf{d})$  correspond bijectively to the iso-classes  $[V(x)]$  of finite dimensional representations of  $\mathcal{Q}$  with the dimension vector  $\mathbf{d}$ . The stabilizer  $GL(\mathbf{d})_x = \{g \in GL(\mathbf{d}) \mid gx = x\}$  of  $x$  is the group of automorphisms of  $M := V(x)$  which is zariski-open in  $\text{End}_{A_n\text{-mod}}(M)$  and has a dimension equal to the  $\dim_{A_n\text{-mod}}(M)$ . It follows that the orbit  $\mathcal{O}_M := \mathcal{O}_x$  of  $M$  has a dimension

$$\dim \mathcal{O}_M = \dim GL(\mathbf{d}) - \dim \text{End}_{A_n\text{-mod}}(M).$$

Now we have the following result, whose proof is the same as the one in [24].

**Lemma 4.1.** *For  $x \in R(\mathbf{d}_1)$  and any  $y \in R(\mathbf{d}_2)$ , let  $\mathcal{E}(\mathcal{O}_x, \mathcal{O}_y)$  be the set of all  $z \in R(\mathbf{d})$  where  $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$  such that  $V(z)$  is an extension of some  $M \in \mathcal{O}_x$  by some  $N \in \mathcal{O}_y$ . Then  $\mathcal{E}(\mathcal{O}_x, \mathcal{O}_y)$  is irreducible.*

□

Given any two finite-dimensional representations of  $M, N$  of the infinite linear quiver  $A_\infty$ , let us consider the extensions

$$0 \longrightarrow N \longrightarrow L \longrightarrow M \longrightarrow 0$$

of  $M$  by  $N$ . By the lemma, there is a unique (up to isomorphism) such extension  $G$  with  $\dim \mathcal{O}_G$  being maximal. We call  $G$  the generic extension of  $M$  by  $N$ , and denoted by  $M * N$ . For any two representations  $M, N$ , we say  $M$  degenerates to  $N$ , or that  $N$  is a degeneration of  $M$ , and write  $[N] \leq [M]$  (or simply  $N \leq M$ ) if  $\mathcal{O}_N \subseteq \overline{\mathcal{O}_M}$  which is the closure of  $\mathcal{O}_M$ . Note that  $N < M$  if and only if  $\mathcal{O}_N \subset \overline{\mathcal{O}_M} \setminus \mathcal{O}_M$ .

Similar to the result in [9, 24], one knows that the relation  $\leq$  is independent of the base field  $k$  and it provides a partial order on the set  $\Lambda$  via setting  $\lambda \leq \mu$  if and only if  $M_k(\lambda) \leq M_k(\mu)$  for any given algebraically closed field  $k$ .

Using the same arguments as in [9, 24], we shall have the following result.

**Theorem 4.1.** (1). *If  $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$  is exact and non-split, then  $M \oplus N < E$ .*

(2). *Let  $M, N, X$  be finite dimensional representations of the quiver  $A_\infty$ . Then  $X \leq M * N$  if and only if there exist  $M' \leq M, N' \leq N$  such that  $X$  is an extension of  $M'$  by  $N'$ . In particular, we have  $M' \leq M, N' \leq N \implies M' * N' \leq M * N$ .*

(3). *Let  $\mathcal{M}$  be the set of isoclasses of finite dimensional representations of  $A_\infty$  and define a multiplication  $*$  on  $\mathcal{M}$  by  $[M] * [N] = [M * N]$  for any  $[M], [N] \in \mathcal{M}$ . Then  $\mathcal{M}$  is a monoid with identity  $1 = [0]$  and the multiplication  $*$  preserves the induced partial ordering on  $\mathcal{M}$ .*

(4).  *$\mathcal{M}$  is generated by irreducible representations  $[S_i], i \in I$  subject to the following relations*

- (1)  $[E_i] * [E_j] = [E_j][E_i]$  if  $i, j$  are not connected by an arrow,
- (2)  $[E_i] * [E_j] * [E_i] = [E_i] * [E_i] * [E_j]$  and  $[E_j] * [E_i] * [E_j] = [E_i] * [E_j] * [E_j]$  if there exists an arrow from  $i$  to  $j$ .

□

In addition, let us denote by  $U_q^+$  the  $\mathbb{Q}[q]$ -algebra generated by  $E_i, i \in I$  subject to the relations

- (1)  $E_i E_j = E_j E_i$  if  $i, j$  are not connected by an arrow,
- (2)  $E_i^2 E_j - (q+1)E_i E_j E_i + qE_j E_i^2 = 0$  and  $E_i E_j^2 - (q+1)E_j E_i E_j + qE_j^2 E_i = 0$  if there exists an arrow from  $i$  to  $j$ .

Let  $H_q(A_\infty)$  denote by the generic Ringel-Hall algebra defined over  $\mathbb{Q}[q]$  associated to the infinite quiver  $A_\infty$ .

Adopting the same argument in [24], we shall further have the following result.

**Theorem 4.2.** *The following algebras are isomorphic.*

- (1) *The monoid ring  $\mathcal{M}$ .*



- (2) The  $\mathbb{Q}$ -algebra with generators  $i \in I$  and relations
  - (a)  $ij = ji$  if  $i$  and  $j$  are connected by arrow,
  - (b)  $iji = iij$  and  $jij = ijj$  if there is an arrow from  $i$  to  $j$ .
- (3) The specialization  $U_0^+$  of  $U_q^+$  to  $q = 0$ .
- (4) The specialization  $H_0(A_\infty)$  of  $H_q(A_\infty)$  to  $q = 0$ .

□

Let us denote by  $\Omega$  the set of all words formed by letters in  $I$ . It is easy to see that for any given word  $w = w_1 \cdots w_m \in \Omega$ , we can set the following finite dimensional representations of  $A_\infty$

$$M(w) = S_{w_1} * S_{w_2} * \cdots * S_{w_m}.$$

Note that there is a unique  $M(\mathfrak{p}(w)) \in A_\infty\text{-mod}$  such that  $M(w) \cong M(\mathfrak{p}(w))$ , which enables us to define a function as follows

$$\mathfrak{p}: \Omega \longrightarrow A_\infty\text{-mod}, w \mapsto M(\mathfrak{p}(w)).$$

Furthermore, we shall have the following result on this function.

**Theorem 4.3.** *The map  $\mathfrak{p}$  induces a surjection*

$$\mathfrak{p}: \Omega \longrightarrow A_\infty\text{-mod}.$$

**Proof:** Once again, we can restrict the function to a certain subcategory  $A_m\text{-mod}$ , where the property holds. □

Therefore,  $\mathfrak{p}$  induces a partition of the set  $\Omega = \cup_{\lambda \in \Lambda} \Omega_\lambda$  with  $\Omega_\lambda = \mathfrak{p}^-(\lambda)$ . We will call each  $\Omega_\lambda$  a fiber of the map  $\mathfrak{p}$ .

Now we are going to recall some information on the partial ordering  $\leq$ . Let  $w = i_1 \cdots i_m$  be a word in  $\Omega$ . Then  $w$  can be uniquely expressed in the tight form  $w = j_1^{e_1} \cdots j_t^{e_t}$  where  $e_r \geq 1, 1 \leq r \leq t$ , and  $j_r \neq j_{r+1}$  for  $1 \leq r \leq t-1$ . A filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

of a module  $M$  is called a reduced filtration of type  $w$  if  $M_{r-1}/M_r \cong e_r S_r$ , for all  $1 \leq r \leq t$ . Note that any reduced filtration of  $M$  of type  $w$  can be refined to a composition of  $M$  of type  $w$ . Conversely, given a composition series of  $M$ , there is a unique reduced filtration of  $M$ . Let us denote by  $\varphi_w^\lambda(x)$  the Hall polynomial  $\varphi_{\mu_1 \cdots \mu_t}^\lambda(x)$  where  $M(\mu_r) = e_r S_r$ . Let us denote by  $\gamma_w^\lambda(q_k)$  the number of the reduced filtrations of  $M_k(\lambda)$  over the base field  $k$  when  $k$  is a finite field. A word  $w$  is called distinguished if  $\gamma_w^{\mathfrak{p}(w)} = 1$ . Note that  $w$  is distinguished if and only if, for some algebraically closed field  $k$ ,  $M_k(\mathfrak{p}(w))$  has a unique reduced filtration of type  $w$ . Similar to [9], we have the following results.

**Lemma 4.2.** (See also **Lemma 4.1** in [9]) Let  $\omega \in \Omega$  and  $\mu \geq \lambda$  in  $\Lambda$ . Then  $\varphi_\omega^\mu \neq 0$  implies  $\varphi_\omega^\lambda \neq 0$ .

□

**Theorem 4.4.** (See also **Theorem 4.2** in [9]) Let  $\lambda, \mu \in \Lambda$ . Then  $\lambda \leq \mu$  if and only if there exists a word  $\omega \in \mathfrak{p}^{-1}(\mu)$  such that  $\varphi_\omega^\lambda \neq 0$ .

□

**Lemma 4.3.** (See also **Lemma 5.2** in [9]) Every fiber of  $\mathfrak{p}$  contains a distinguished word.

□

Let us define  $[[e_a]]^! = [[1]] \cdots [[e_a]]$  with  $[[m]] = \frac{1-(rs^{-1})^m}{1-rs^{-1}}$ . Then we shall have the following result.

**Lemma 4.4.** (See also **Lemma 6.1** in [9]) Let  $w \in \Omega$  be a word with the tight form  $j_1^{e_1} \cdots j_t^{e_t}$ . Then, for each  $\lambda \in \Lambda$ ,

$$\varphi_w^\lambda(rs^{-1}) = \gamma_w^\lambda(rs^{-1}) \prod_{r=1}^t [[e_r]]^!.$$

In particular,  $\varphi_w^{\mathfrak{p}(w)} = \prod_{r=1}^t [[e_r]]^!$  if  $w$  is distinguished.

□

For any given word  $w = i_1 \cdots c_m \in \Omega$ , we can associate a monomial

$$u_w = u_{i_1} \cdots u_{i_m} \in H_{r,s}(A_\infty).$$

**Proposition 4.1.** For any  $w \in \Omega$  with the tight form  $j_1^{r_1} \cdots j_t^{e_t}$ , we have

$$u_w = \sum_{\lambda \leq \mathfrak{p}(w)} \varphi_w^\lambda(rs^{-1}) u_\lambda = \prod_{r=1}^t [[e_r]]^! \sum_{\lambda \leq \mathfrak{p}(w)} \gamma_w^\lambda(rs^{-1}) u_\lambda.$$

Moreover, the coefficients appearing in the sum are all nonzero.

□

As a result, we shall have the following theorem.

**Theorem 4.5.** For each given  $\lambda \in \Lambda$ , let us choose an arbitrary word  $w_\lambda \in \mathfrak{p}^{-1}(w)$ . Then the set  $\{u_{w_\lambda} \mid \lambda \in \Lambda\}$  is a  $\mathbb{Q}(r, s)$ -basis of  $H_{r,s}(A_\infty)$ . Moreover, if all the words are chosen to be distinguished, then this set is a  $\mathbb{Z}[r, s]_{(r-1, s-1)}$ -basis of  $H_{r,s}(A_\infty)_{\mathbb{Z}[r, s]_{(r-1, s-1)}}$ .

□

**4.1. A bar-invariant basis of  $U_{r,s}^+(\mathfrak{sl}_\infty)$ .** It is easy to see that the algebra  $U_{r,s}^+(\mathfrak{sl}_\infty)$  admits a  $\mathbb{Q}$ -linear involution defined as follows

$$\bar{r} = s, \bar{s} = r, \bar{e}_i = e_i \text{ for all } i \in I.$$

And we will refer this involution as the bar-involution. In this subsection, we will construct a bar-invariant basis for  $U_{r,s}^+(\mathfrak{sl}_\infty)$ .

Denote by  $[M, N] = \dim_k \text{Hom}(M, N)$  and  $[M, N]^1 = \dim_k \text{Ext}^1(M, N)$ . Let us set  $c_{M,N}^X = s^{[X,X]-[M,N]+[M,N]^1-[M,M]-[N,N]} F_{M,N}^X(rs^{-1})$ . It is obvious that the same proof in [24] shall yield the following result.

**Proposition 4.2.** *Let us write  $\bar{u}_\alpha = \sum_\beta \omega_\beta^\alpha u_\beta$ , then we have*

- (1)  $\omega_\beta^\alpha = 0$  unless  $\beta \leq \alpha$ , and  $\omega_\alpha^\alpha = 1$ ,
- (2) if  $u_\alpha = M \oplus N$  for finite dimensional representations  $M, N$  with  $[N, M] = 0 = [M, N]^1$ , then

$$\omega_\beta^\alpha = \sum_{M' \leq M, N' \leq N} \omega_{M'}^M \omega_{N'}^N c_{M',N'}^\alpha,$$

- (3) if  $u_\alpha$  is an exponent of a finite dimensional indecomposable representation, then

$$\omega_\beta^\alpha = s^{[u_\beta, u_\beta]^1} - \sum_{\beta \leq \gamma < \alpha} r^{[u_\gamma, u_\gamma]^1} \omega_\beta^\gamma,$$

- (4)  $\omega_\beta^\alpha \in s^{[u_\beta, u_\beta]-[u_\alpha, u_\alpha]} \mathbb{Z}[rs^{-1}]$ .

□

Furthermore, using the arguments in [21, 24], we shall have the following result on a bar-invariant basis of the algebra  $U_{r,s}^+(\mathfrak{sl}_\infty)$ .

**Theorem 4.6.** *For each isoclass  $\alpha$ , there exists a unique element*

$$\mathcal{C}_\alpha \in u_\alpha + s^{-1} \mathbb{Z}[rs, r^{-1}s^{-1}, s][\mathbf{B} \setminus \{u_\alpha\}]$$

*such that  $\bar{\mathcal{C}}_\alpha = \mathcal{C}_\alpha$ . Write  $\mathcal{C}_\alpha = \sum_\beta \zeta_\beta^\alpha u_\beta$ , we have*

- (1)  $\zeta_\beta^\alpha = 0$  unless  $\beta \leq \alpha$ , and  $\zeta_\alpha^\alpha = 1$ ,
- (2)  $\zeta_\beta^\alpha \in s^{[u_\beta, u_\beta]-[u_\alpha, u_\alpha]} \mathbb{Z}[rs^{-1}]$ ,
- (3) Denote by  $\hat{\zeta}_\beta^\alpha(v) \in \mathbb{Z}[v, v^{-1}]$  the specialization of  $\zeta_\beta^\alpha$  to  $\alpha = v = s^{-1}$ , we have

$$\zeta_\beta^\alpha = (\sqrt{rs})^{[u_\beta, u_\beta]-[u_\alpha, u_\alpha]} \hat{\zeta}_Y^X(\sqrt{rs^{-1}}).$$

□

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